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Application of Black-Scholes-Merton Model in Option Pricing and Intangibles Assets

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Abstract

The Black-Scholes model was developed by Fisher Black and Myron Scholes in the 1970s to price stock options. Since then the model has been suited to price so-called intangible assets such as trademarks and patents. In this paper, we investigate the related Black-Scholes-Merton model and the relevant characteristics of patents in order to associate patents as real options. After describing patents as options, we apply the Black-Scholes-Merton model to the valuation of the intangible assets. Special attention is given to modeling volatility and the cost of delay in order to obtain the best patent price and the optimal time to commercialize the patent. Finally, we apply our patent price model to study the case of an upcoming Apple product.

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I. Introduction:

In the finance world, quantitative methods and pricing models are the fields that many analysts attempt to master in order to win in the market game. The way the world uses mathematical models to predict and evaluate derivatives price has been changing and advancing every day.

In 1973, Fischer Black, Myron Scholes introduced the Black-Scholes model, which eventually became one of the most popular option pricing models. Although there are some limitations, the model is used frequently in the real world until today and has several applications, not just options pricing.

II. Put/Call Options and Black-Scholes Model:

An option of a security is the right, but not the obligation, to buy or sell an amount of the underlying security at a pre-determined strike price. With the power to limit the risk, options are useful in managing portfolios to secure profit, or as a hedging strategy against the market movements. While call options give buyers the right to buy an underlying security, put options give buyers the right to sell. Each option has a purchase price called premium, a strike price and an expiration date. The option prices vary based on the risk of the security, strike price and time to expiration.

There are two types of options: American options and European options. American options allow holders to execute the options any time before the expiration date while European options can only be executed on the expiration date. Due to this flexibility, American options cannot be priced by the same method as European and there are no general formula to price this asset. [1]

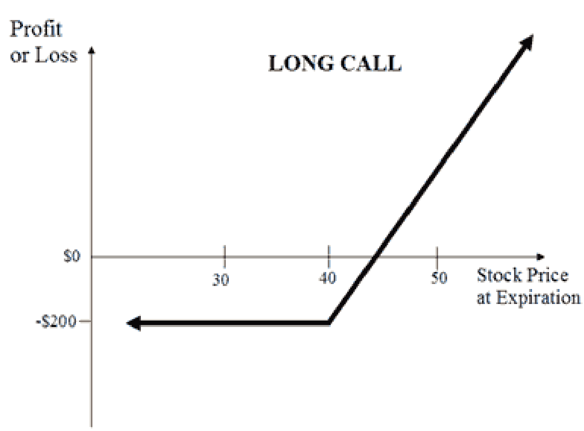

Figure 1: Long call profit or loss graph (theoptionguides.com)

[1] Grossinho, Maria do Rosário, Yaser Faghan Kord, and Daniel Sevcovic. "Pricing American Call Options by the Black-Scholes Equation with a Nonlinear Volatility Function." *arXiv preprint arXiv:1707.00358* (2017).

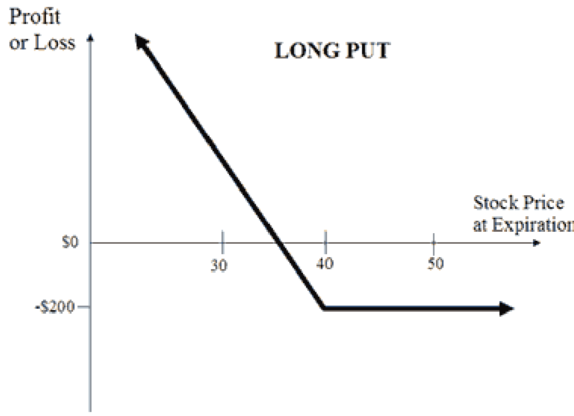

Figure 2: Long put profit or loss graph (theoptionguides.com)

In an attempt to determine an appropriate price for European options, many financial models have been developed and the most popular model is the Black-Scholes model (BS). BS is a partial differential equation derived from geometric Brownian motion by Ito's Lemma and stochastic calculus. Geometric Brownian motion is a continuous-time stochastic process following a Wiener process that describes the movement of a random walk (similar to the trajectory of a stock price).

The BS makes some assumptions:

1. The option is only exercised at expiration (European option).
2. The stock pays no dividend during the lifetime of an option (Dividend makes no effect on the valuation of the option).
3. Efficient market (market stock prices follow geometric Brownian motion).
4. Risk-free interest rate is constant during the lifetime of the option.
5. Known and constant volatility.
6. The returns are lognormally distributed (only positive outcome).
   Lognormal distribution is derived from normal distribution by using logarithmic mathematics. The distribution is right-skewed as it only consider positive outcome.

A derivation of the Black-Scholes formula (the solution of the stochastic differential equation) can be found in many sources. For example, the process of deriving is proven by Martin Haugh[2]. Assume stock price follows a random walk, Let S_t denote the price of the stock at time t. According to assumptions 3 and 6, S_t satisfies the stochastic differential equation [] where W_t is a Brownian process

[2] Haugh, M., Prof. (n.d.). *The Black-Scholes Model*. Lecture notes. Retrieved July 5, 2019, from https://pdfs.semanticscholar.org/53c7/e65c573693f6f10e06ab7a91fe09c7881f28.pdf

$$dS_t = \mu S_t\, dt + S_t \sigma\, dW_t$$

while μ is the drift in the Geometric Brownian motion and σ is the volatility of the underlying security.

By using a self-financing trading strategy and Itô's Lemma from stochastic calculus, the corresponding solution takes the form[3]:

$$\boldsymbol{C(S_t, t) = S_t N(d_1) - e^{-rT} K N(d_2)}$$

when:

$$d_1 = \frac{ln\,(\frac{S_t}{K}) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}$$

$$d_2 = d_1 - \sigma\sqrt{T - t}$$

And:

S: the current price of the stock/security.
K: the strike or exercise price of the option.
r: annual risk-free interest rate
T: time till time of expiration
σ: Volatility of the underlying asset; for example, the standard deviation of a stock's annual returns
N: standard normal cumulative distribution function

Along with the computation of the option price, The BS formula can also be used to compute the *Greeks*, the parameters of sensitivity of an option.

1. Delta: The sensitivity of option price to the change in the price of the underlying security.

$$\Delta = \frac{dC}{dS} = e^{-qT} N(d_1).$$

2. Gamma: The sensitivity of delta of the option to the change in the price of the underlying security.

$$\Gamma = \frac{d^2C}{dS^2} = e^{-qT} \frac{N(d_1)}{\sigma S \sqrt{T}}.$$

3. Vega: the sensitivity of an option to the change in volatility.

$$\nu = \frac{dC}{d\sigma} = e^{-qT} S\sqrt{T} N(d_1).$$

4. Theta: The sensitivity of an option to the change in time-to-maturity.

[3] Evans, Lawrence C., *An Introduction to Stochastic Differential Equations*, 2013, American Mathematical Society, Providence R.I. and cite Ross, Sheldon M., *An Elementary Introduction to Mathematical Finance*, 2nd ed., 2003, Cambridge University Press, Cambridge

$$\Theta = -\frac{dC}{dT} = -e^{-qT}SN(d_1)\frac{\sigma}{2\sqrt{T}} + qe^{-qT}SN(d_1) - rKe^{-rT}N(d_2).$$

5. Rho: The sensitivity of an option price to the change in interest rate

$$\rho = \frac{dC}{dr}.$$

III. Describing patents as options

1. Patents & options:

A patent is a document which describes an invention. After a patent application is approved by a government office (or a regional office acting for several countries) a patent creates a legal situation in which the patented invention can normally only be exploited (manufactured, used, sold, imported) with the authorization of the owner of the patent[4]. Different from copyrights, which refer mostly to an expression of ideas, artwork or music, patents refer to the invention of a product that can be exploited for business.[5]

In the United States, a patent's lifetime is 20 years, starting from the filing date.

Holding a patent gives the owner the right to develop the invention legally. For example, a patent of iOS allows Apple to manufacture products using the technology and similar accessories using the platform. The owner of a patent has the right to decide whether or not to publish the invention and start to generate income from its business. After investing an initial cost of developing the invention, the owner can run the business to receive income. If not, the maximum loss is the initial investment and the patent filing cost. This characteristic is similar to a call option which allows the holder to buy shares of stock at the strike price. If the holder decides not to execute the option, the maximum loss is the premium paid for the option. Therefore, patents can be described as an option or a real option, which operates in a similar manner, referring to managerial decisions rather than the buy/sell decisions of regular options.[6] The owners of the patent have the right to choose whether or not to initiate the project and start to generate an income stream. If not, they have a right to postpone the patent process until the optimal time to publish the invention.[7]

2. Modified Black-Scholes-Merton model for patent pricing:

Comparing the characteristics of the BS model to patents, the original BS model lacks some components and variables that are different from the nature of patents.

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[4] "Chapter 2 - Fields of Intellectual Property ." *WIPO Intellectual Property Handbook: Policy, Law and Use*, WIPO PUBLICATION, 2004, pp. 17–18.

[5] "The Difference Between Copyright and Patent." *CJAM*, 21 Oct. 2014, cjam.info/en/difference-copyright-and-patent/.

[6] Cotropia, Christopher A. "Describing Patents as Real Options", 34 J. Corp. L. 1127 (2009).

[7] Kramná, Eva. "Economic Valuation of Patents as Real Options." *Finance and the Performance of Firms in Science, Education and Practice* (2011).

Therefore, to use the model in patents or intangibles asset pricing, the BS has to be modified to accurately estimate the asset's price.

The more suitable model is the closely related Black-Scholes-Merton (BSM) model. It was applied to pricing intangible assets by new modified model introduced by Dr. Aswath Damodaran of NYU.[8]

$$\boldsymbol{C(S_t, t) = S_t e^{-yT} N(d_1) - e^{-rT} K N(d_2)}$$

Where:

$$d_1 = \frac{ln\,(\frac{S_t}{K}) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

C: option price
S: Net present value (NPV) of future cash flows from manufacturing the product.
K: NPV of cost of developing the product.
T: time till time of expiration
r : annual risk-free interest rate
σ: Volatility of the underlying asset
N: standard normal cumulative distribution function
y: annual dividend rate for stocks paying dividends, and the cost of delay for patents

The price of a patent at a specific time is determined based on its net present value (NPV) of future profit it will generate in the remainder of its lifetime. For example, if the owner of a patent published the product at *t = 0*, the value of the patent will be estimated by all the possible future cash flows less any initial investment costs . However, as *t* varies, the price of patent is affected by a cost of delay *t* years after the filing date. As *t* is positively correlated with the cost of delay, the longer the owner postpones the project, the lower value of the patent will be. This depreciation in value agrees with market forces such as competition from rival companies

[8] Damodaran, Aswath. "The Value of Intangibles." *Online version available at http://pages. stern. nyu. edu/~ adamodar/pdfiles/damodaran2ed/ch12. pdf (viewed on 28th October 2015)* (2008).

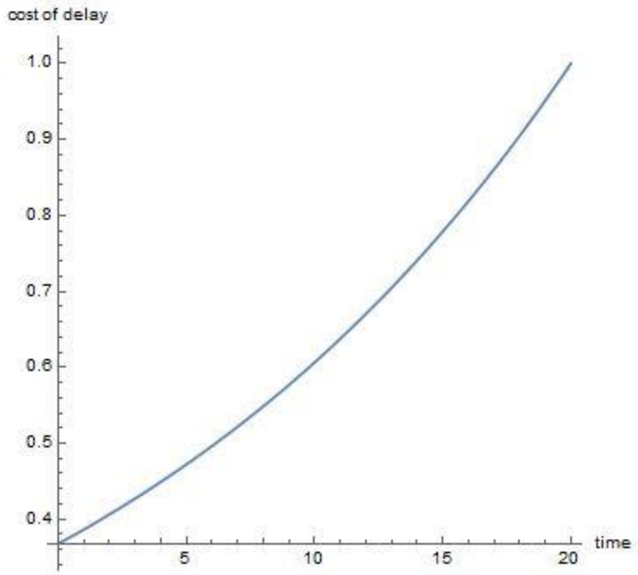

Figure 3: Cost of delay

IV. Application to pricing a patent

In the application part of this project, we aim to determine the price of Apple Airpods patents. In 2016, Apple filled in several patents to cover its new wireless *Airpods* with charging case.[9] The product is listed in Apple's accessories and is considered one of the main product lines that provides an enormous income stream for the company. The first model is on sale with the price of $159 for a pair of earbuds and one charging case. The newest version is with wireless charging case. In this section we estimate the price of the first version of *Airpods*, assuming the line will not be shut down by Apple before the expiration date.

1. Data input:

Apple profit margin	0.35
Percentage of Equity	
Risk-free rate (20-year government bond)	2.21%

[9] "US20170094394A1 - Earbud Case with Charging System." *Google Patents*, Google, patents.google.com/patent/US20170094394A1/en?oq=apple%2Bcharging%2Bcase.

Market rate (S&P500)	7.00%
Apple most recent interest	3240
Average debt 2017- 2018	115081
Decline rate of sale[10]	0.4
Tax rate	24.6%

Table 1 [11]

2. Weighted average cost of capital (WACC) calculation:

% Equity (CAPM)	88.95%
Risk-free rate (10-year treasury)	2.21%
Beta	1.09
Market rate	7%
Cost of equity	**9.93%**
% Debt	0.1105
Interest (millions)	3240
Average debt (millions)	115081
Cost of debt	2.82%

[10] After the mature period of a product life cycle, the product sales start to decline because of new technology or being replaced by other competitors. The product decline rate of 40% is an assumption.
[11] Data is collected and analyzed from Apple Annual Report 2018

Tax rate	24.60%
After tax cost of debt	**2.12%**
WACC	**8.987%**

Table 2

3. Forecasting *Airpods* sales during the patent lifetime:

	2016	2017	2018	2019	2020	2021	2022	2023	2035	2036
n	20	19	18	17	16	15	14	13	1	0
Sale (unit)		16	35	55	80	110	120	72	0.16	0.09
Sale ($)		$2,560	$5,600	$8,800	$12,800	$17,600	$19,200	$11,520	$25.1	$15
WACC		10.16%								
S	**$59,864**	**$59,864**	**$62,684**	**$62,717**	**$59,554**	**$52,106**	**$39,189**	**$23,511**	**$352**	**$137**
r		2.21%								
COGS		$16,640	$36,400	$57,200	$83,200	$114,400	$124,800	$74,880	$163	$98
K	**$368,250**	**$368,250**	**$389,026**	**$392,153**	**$374,797**	**$329,678**	**$248,774**	**$149,251**	**$229**	**$89**
Profit ($)		$16,640	$56,000	$88,000	$128,000	$176,000	$192,000	$115,200	$251	$150

Table 3 (In millions)[12]

[12] See Appendix I for further details.

In table 3, *S* is the net present value of all future cash flows generated by the exploitation of the patent while *K* is the NPV of cost of goods sold. The forecast of sales from 2017 to 2019 is based on the research note by Ming-Chi Kuo of TF International Securities[13].

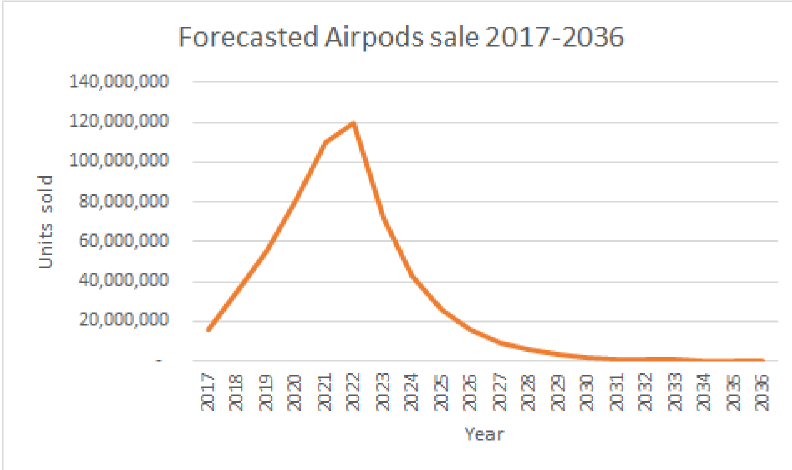

Figure 4: Forecasted *Airpods* sales

On March 2019, Apple released the new version of *Airpods* with wireless charging case with an upgraded H-1 Chip and better connectivity with the earbuds. The cost for the updated product is $199 for a pair of earbuds and wireless charging case. This product is compatible with any Qi charging mat. This new version will eventually slow down and take over the sales of the first version. After the maturity stage, the sale of the product is expected to decline rapidly at the annual rate of 40% (assumption).

4. **Estimate the change of volatility of Apple**

According to the assumptions of Black-Scholes model, volatility of the stock is known and constant throughout the lifetime of the option. Since Apple stock is the representative of the company's market value, it is realistic to use an estimate ofthe 10-year historical volatility of Apple's stock (symbol: APPL) in the calculation of a patent owned by Apple. Using AAPL stock closing prices available online (e.g. finance.yahoo.com) we determined the annualized volatility came out to σ = 26.22%.

5. **Patent pricing by Black Scholes Model:**

S	K	σ	r	y	t	C
$59,863,512,059	$38,911,282,838	26.23%	2.21%	5.00%	2001.37%	**$8,959,130,205**
$59,863,512,059	$38,911,282,838	26.23%	2.21%	5.00%	1901.37%	**$9,354,570,624**

[13] Gillespie, Emily. "Analyst Says *Airpods* Sales Will Go Through the Roof Over the Next Few Years, Report Says." *Fortune*, Fortune, 4 Dec. 2018, fortune.com/2018/12/03/analyst-says-*Airpods*-sales-will-go-through-the-roof-over-the-next-few-years-report-says/.

$62,683,596,367	$40,744,337,639	26.23%	2.21%	5.00%	1801.37%	**$10,224,681,326**
$62,717,128,741	$40,766,133,682	26.23%	2.21%	5.00%	1701.37%	**$10,675,352,727**
$59,553,674,754	$38,709,888,590	26.23%	2.21%	5.00%	1601.10%	**$10,575,707,788**
$52,105,913,206	$33,868,843,584	26.23%	2.21%	5.00%	1501.10%	**$9,649,035,445**
$39,188,802,605	$25,472,721,693	26.23%	2.21%	5.00%	1401.10%	**$7,564,534,365**
$23,510,798,804	$15,282,019,223	26.23%	2.21%	5.00%	1301.10%	**$4,728,485,583**
$14,103,773,392	$9,167,452,705	26.23%	2.21%	5.00%	1200.82%	**$2,954,374,684**
$8,459,314,960	$5,498,554,724	26.23%	2.21%	5.00%	1100.82%	**$1,844,443,604**
$5,072,374,860	$3,297,043,659	26.23%	2.21%	5.00%	1000.82%	**$1,150,515,256**
$3,039,921,939	$1,975,949,260	26.23%	2.21%	5.00%	900.82%	**$716,839,241**
$1,820,135,365	$1,183,087,987	26.23%	2.21%	5.00%	800.55%	**$445,954,204**
$1,087,920,305	$707,148,199	26.23%	2.21%	5.00%	700.55%	**$276,725,280**
$648,217,318	$421,341,257	26.23%	2.21%	5.00%	600.55%	**$171,045,983**
$383,987,966	$249,592,178	26.23%	2.21%	5.00%	500.55%	**$105,038,503**
$225,006,166	$146,254,008	26.23%	2.21%	5.00%	400.27%	**$63,783,561**
$129,132,978	$83,936,436	26.23%	2.21%	5.00%	300.27%	**$37,942,306**
$71,081,449	$46,202,942	26.23%	2.21%	5.00%	200.27%	**$21,705,198**
$35,675,496	$23,189,073	26.23%	2.21%	5.00%	100.27%	**$11,446,437**
$13,805,211	$8,973,387	26.23%	2.21%	5.00%	0.27%	**$4,830,476**

Table 4

Table 4 shows the results of the *Airpods* patent price each year as calculated by Black-Scholes-Merton model. Recall that the BSM model is a modified Black-Scholes model that does account for dividends, included here as the modified model for patent pricing included $\frac{1}{20} = 5\%$ cost of delay.

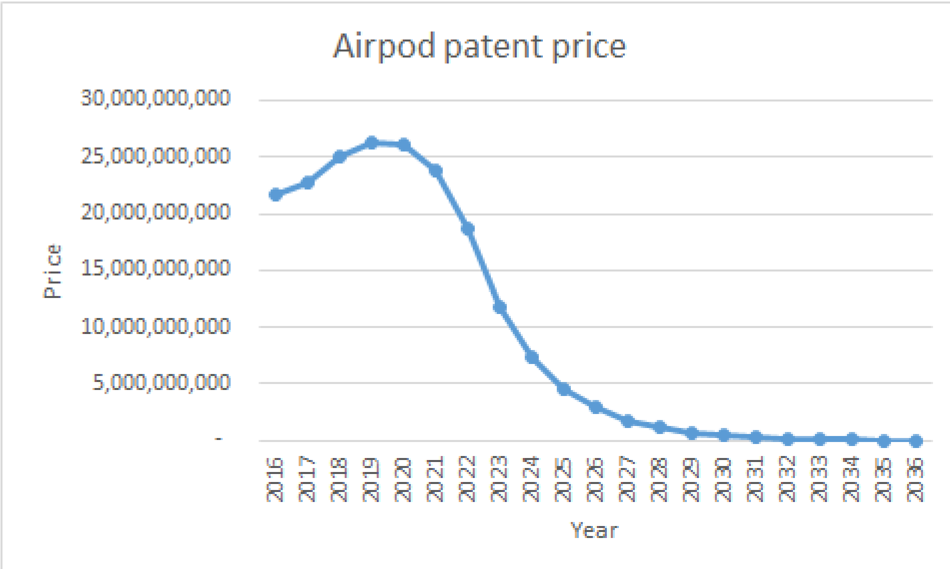

Figure 5

Figure 5 illustrates the value of the Airpods patent throughout its lifetime. The patent value is at its highest in 2019. It is the same time when Apple introduced the new wireless *Airpods* 2 with several new patents filed in for the new design and technology. The patents filed for the new model will devalue the patents of the first, as the second model sale will eventually replace the sales of the first model. Therefore, the time the first patent hits its highest price is when the second model came out.

6. Discussion:

The Black-Scholes-Merton model provides the estimated valuation of a patent accommodating the price movement with the effect of project delay on the price. It can also be used to evaluate the price of other intangible assets such as goodwill, brand name or copyrights. The idea is that based on the characteristic of the asset, we can forecast the future cash flow and estimate the net present value of the asset by a discounted cash flow model. For such cases, we can use the Black-Scholes-Merton model to compute the present value of the intangible asset as an option.

However, the model also have some limits in estimating the price of options/assets. In the preceding example, the volatility is constant and known throughout the whole lifetime of the option. This assumption is definitely incorrect in real life. A company's volatility is always fluctuating and therefore never a constant. The concept of implied volatility, which arises in options pricing, can be used to produce a surface that usually contains a so-called "volatility smile" which indicates that the distribution of options with the same time to expiration will have a shape of a smile. In most cases, the distribution is skewed as in Figure 6[14]. Black-Scholes assumes that volatility is constant and remains so throughout the lifetime of the option. This in-turn would produce a volatility surface that is flat over time.

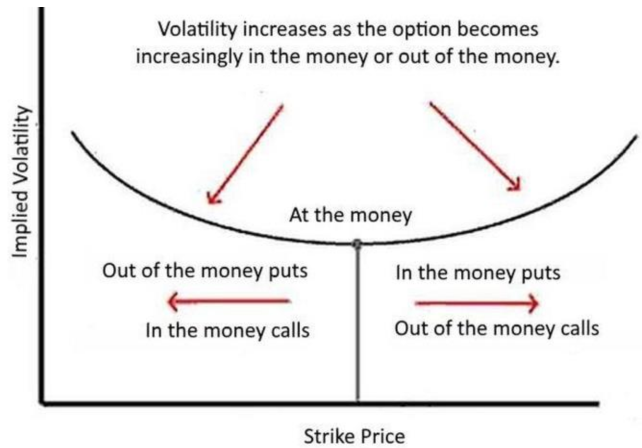

Figure 6: Volatility smile - investopedia.com

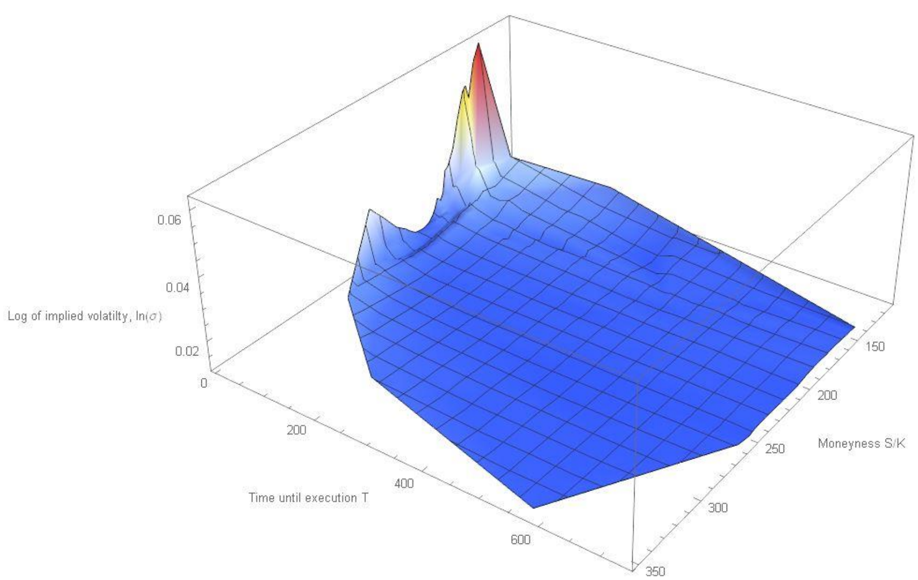

Figure 7: Apple's Implied Volatility surface derived from Apple's 0-180-day options

Figure 7 is the implied volatility surface extracted from Apple's most recent options data. The implied volatility surface for the stock AAPL shows the actual value of the volatility parameter σ needed for the Black-Scholes formula to agree with the actual

[14] Butler, Brian Michael 1969-, "The Black-Scholes formula and volatility smile." (2012). *Electronic Theses and Dissertations.* Paper 188.

market price of the option. Hence, in general, volatility is not constant at any time of the option’s lifetime.

7. Conclusion:

The well-known BSM model has been practiced and became popular for its applications in several fields. Although there are some flaws in the methodology, for example, the model cannot resemble accurately the market movement, it still provides a close and reliable valuation method in pricing options and other assets that share its core qualities.

As the world is advancing, there are a lot of modifications of the Black-Scholes model to better reflect the market movement and more accurate pricing.

APPENDIX I: 20 years sales forecasting of Airpods (in millions)

	2016	2017	2018	2019	2020	2021	2022	2023	2024	2025	2026
n		19	18	17	16	15	14	13	12	11	10
Sale (unit)		16	35	55	80	110	120	72	43	26	16
Sale ($)		$2,560	$5,600	$8,800	$12,800	$17,600	$19,200	$11,520	$6,912	$4,147	$2,488
WACC		8.987%									
S	$59,864	$59,864	$62,684	$62,717	$59,554	$52,106	$39,189	$23,511	$14,103	$8,459	$5,072
r		2.21%									
COGS ($)		$1,664	$3,640	$5,720	$8,320	$11,440	$12,480	$7,488	$4,493	$2,696	$1,617
K	$38,911	$38,911	$40,744	$40,766	$38,710	$33,869	$25,473	$15,282	$9,167	$5,499	$3,297

	2027	2028	2029	2030	2031	2032	2033	2034	2035	2036
n	9	8	7	6	5	4	3	2	1	0
Sale (unit)	9	6	3	2	1	1	0	0.26	0.16	0.09
Sale ($)	1,493	896	537	322	193	116	70	42	25	15
WACC		8.987%								
S	$3,039.92	$1,820.14	$1,087.92	$648.22	$383.99	$225.01	$129.13	$71.08	$35.68	$13.81
r		2.21%								
COGS ($)	$970	$582	$349	$210	$126	$75	$45	$27	$16	$10
K	$1,976	$1,183	$707	$421	$250	$146	$84	$46	$23	$9